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On a class of degenerate and singular elliptic systems in bounded domains

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ABSTRACT

This paper deals with the nonexistence and multiplicity of nonnegative, nontrivial solutions to a class of degenerate and singular elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda F_v(x,u,v) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^N$, $N \geqq 2$, and $h_i : \Omega \to [0,\infty)$, $h_i \in L^1_{loc}(\Omega)$, h_i $(i = 1,2)$ are allowed to have "essential" zeroes at some points in Ω, $(F_u, F_v) = \nabla F$, and λ is a positive parameter. Our proofs rely essentially on the critical point theory tools combined with a variant of the Caffarelli–Kohn–Nirenberg inequality in [P. Caldiroli, R. Musina, On a variational degenerate elliptic problem, NoDEA Nonlinear Differential Equations Appl. 7 (2000) 189–199].

1. Introduction and main results

In this paper, we are concerned with a class of semilinear elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla u) = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$, $(F_u, F_v) = \nabla F$ stands for the gradient of F in the variables $w = (u,v) \in \mathbb{R}^2$ and λ is a positive parameter. We point out the fact that if $h_1(x) = h_2(x) \equiv 1$, the problem was intensively studied in the last decades. We refer to some interesting works [3,8,10,17,23].

In a recent paper [6], P. Caldiroli and R. Musina have considered the Dirichlet elliptic problem of the form

$$-\operatorname{div}\big(h(x)\nabla u\big) = f(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where Ω is a (bounded or unbounded) domain in $\mathbb{R}^N$ $(N \geqq 2)$, and h is a nonnegative measurable weighted function that is allowed to have "essential" zeroes at some points in Ω, i.e., the function h can have at most a finite number of zeroes in Ω. More precisely, the authors assumed that there exists an exponent $\alpha \in (0,2]$ such that the function h decreases more slowly than $|x-z|^\alpha$ near every point $z \in h^{-1}\{0\}$. Then, they proved some interesting compact results and obtained the existence of a nontrivial solution for (1.2) in a suitable function space using the Mountain pass theorem [1]. These results were used to

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study the existence of a solution for a class of degenerate elliptic systems by N.B. Zographopoulos [22] and G. Zhang et al. [24].

In [22], N.B. Zographopoulos considered the degenerate semilinear elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda\mu(x)|u|^{\gamma-1}|v|^{\delta+1} & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda\mu(x)|u|^{\gamma+1}|v|^{\delta-1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the functions $h_i \in L^1_{loc}(\Omega)$ and h_i $(i = 1,2)$ are allowed to have "essential" zeroes at some points in Ω, the function $\mu \in L^\infty(\Omega)$ and may change sign in Ω, λ is a positive parameter and the nonnegative constants γ, δ satisfy the following conditions

$$\gamma + 1 < p < 2^\star_\alpha, \qquad \delta + 1 < q < 2^\star_\beta,$$
$$\frac{\gamma+1}{p} + \frac{\delta+1}{q} = 1, \qquad \frac{\gamma+1}{2^\star_\alpha} + \frac{\delta+1}{2^\star_\beta} < 1,$$
$$2^\star_\alpha = \frac{2N}{N-2+\alpha}, \qquad 2^\star_\beta = \frac{2N}{N-2+\beta}, \quad \alpha, \beta \in (0,2).$$

Using arguments of Mountain pass type [1], the author showed the existence of a nontrivial solution of (1.3) in the super-critical case, i.e.

$$\frac{\gamma+1}{2} + \frac{\delta+1}{2} > 1. \tag{1.4}$$

In the critical case $\gamma = \delta = 0$, the author also established the existence of a positive principal eigenvalue λ_1 for system (1.3) and some perturbations of its.

Motivated by the results in [2,6,8,18,22], G. Zhang et al. [24] obtained some existence results for (1.1) under subcritical growth conditions and the primitive $F(x,u,v)$ being intimately related to the first eigenvalue of a corresponding linear system.

In the present paper, we consider system (1.1) with the functions h_i $(i = 1,2)$ as in [22] and [24]. Under the suitable conditions on the nonlinearities $F_u(x,u,v)$ and $F_v(x,u,v)$, using the Minimum principle (see [20, p. 4, Theorem 1.2]) and the Mountain pass theorem of A. Ambrosetti and P. Rabinowitz [1], we show that system (1.1) has at least two nonnegative, nontrivial solutions provided that λ is large enough. We also prove that the system has no nontrivial solution in case when the parameter λ is small enough. Thus, these results are completely natural extensions from [22] and [24]. Our paper is motivated by the interesting ideas introduced in [3,10,13,16]. In order to state our main results, we introduce next some hypotheses on the structure of the problem.

Throughout this paper, we assume the functions h_1 and h_2 satisfying the following conditions:

($\mathbf{H_1}$) The function $h_1 : \Omega \to [0,\infty)$ belongs to $L^1_{loc}(\Omega)$ and there exists a constant $\alpha \geqq 0$ such that

$$\liminf_{x\to z} |x-z|^{-\alpha} h_1(x) > 0 \quad \text{for all } z \in \overline{\Omega}.$$

($\mathbf{H_2}$) The function $h_2 : \Omega \to [0,\infty)$ belongs to $L^1_{loc}(\Omega)$ and there exists a constant $\beta \geqq 0$ such that

$$\liminf_{x\to z} |x-z|^{-\beta} h_2(x) > 0 \quad \text{for all } z \in \overline{\Omega}.$$

It should be observed that a model example for ($\mathbf{H_1}$) (similar to ($\mathbf{H_2}$)) is that $h_1(x) = |x|^\alpha$ (see [11,12]). The case $\alpha = 0$ covers the "isotropic" case corresponding to the Laplacian operator. In [6], the conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) were excellently used by P. Caldiroli and R. Musina. The authors proved that if a function h satisfies the conditions as in ($\mathbf{H_1}$) (similar to ($\mathbf{H_2}$)), then there exist a finite set $Z = \{z_1, z_2, \dots, z_k\} \subset \overline{\Omega}$ and numbers $r, \delta > 0$ such that the balls $B_i = B_r(z_i)$ $(i = 1,2,\dots,k)$ are mutually disjoint and

$$h(x) \geqq \delta|x - z_i|^\alpha \quad \forall x \in B_i,\ i = 1,2,\dots,k,$$

and

$$h(x) \geqq \delta \quad \forall x \in \overline{\Omega} \Big\backslash \bigcup_{i=1}^k B_i.$$

This says the conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) implying that the elliptic operators in system (1.1) are degenerate and singular. Moreover, the sets $Z_{h_1} = \{x \in \overline{\Omega}:\ h_1(x) = 0\}$ and $Z_{h_2} = \{z \in \overline{\Omega}:\ h_2(z) = 0\}$ are finite, the potentials $h_1(x)$ and $h_2(x)$ respectively behave like $|x|^\alpha$ and $|x|^\beta$ around their degenerate points. Such problems come from the consideration of standing waves in anisotropic Schrödinger systems (see [15]). They arise in many areas of applied physics, including nuclear physics, field theory, solid waves and problems of false vacuum. These problems are introduced as models for several physical phenomena related to equilibrium of continuous media which somewhere are perfect insulators (see [9, p. 79]). For more information and connection with problems of this type, the readers may consult in [14,19] and the references therein.

Next, we assume that $F(x,t,s)$ is a C^1-function on $\Omega \times [0,\infty) \times [0,\infty) \to \mathbb{R}$, satisfying the hypotheses below:

($\mathbf{F_1}$) There exist two positive constants C_1 and C_2 such that

$$\left|F_t(x,t,s)\right| \leqq C_1 t^{\gamma} s^{\delta+1}, \qquad \left|F_s(x,t,s)\right| \leqq C_2 t^{\gamma+1} s^{\delta}$$

for all $(t,s) \in \mathbb{R}^2$, a.e. $x \in \Omega$ and some $\gamma, \delta > 1$ with $\frac{\gamma+1}{p} + \frac{\delta+1}{q} = 1$, $\frac{\gamma+1}{2^\star_\alpha} + \frac{\delta+1}{2^\star_\beta} < 1$, and $\gamma + 1 < p < 2^\star_\alpha = \frac{2N}{N-2+\alpha}$, $\delta + 1 < q < 2^\star_\beta = \frac{2N}{N-2+\beta}$, $\alpha, \beta \in (0,2)$.

($\mathbf{F_2}$) There exist positive constants η, s_0, t_0 such that $F(x,t,s) \leqq 0$ for all $(t,s) \in \mathbb{R}^2$ with $t^p + s^q \leqq \eta$ and $F(x,t_0,s_0) > 0$ for a.e. $x \in \Omega$, where p and q are given as in ($\mathbf{F_1}$).

($\mathbf{F_3}$) It holds that

$$\limsup_{|(t,s)|\to\infty,\, t,s>0} \frac{F(x,t,s)}{t^{\gamma+1}s^{\delta+1}} \leqq 0$$

uniformly in $x \in \Omega$.

It is clear that by the presence of the functions h_1, h_2, weak solutions of system (1.1) must be found in a suitable space. To this purpose, we define the Hilbert spaces $H_0^1(\Omega, h_1)$ and $H_0^1(\Omega, h_2)$ as the closures of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\|_{h_1} = \left(\int_\Omega h_1(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}}$$

for all $u \in C_0^\infty(\Omega)$ and

$$\|v\|_{h_2} = \left(\int_\Omega h_2(x)|\nabla v|^2\,dx\right)^{\frac{1}{2}}$$

for all $v \in C_0^\infty(\Omega)$, respectively, and set $H = H_0^1(\Omega,h_1) \times H_0^1(\Omega,h_2)$. Then, it is clear that H is a Hilbert space under the norm

$$\|w\|_H = \|u\|_{h_1} + \|v\|_{h_2}$$

for all $w = (u,v) \in H$, and with respect to the scalar product

$$\langle \varphi, \psi \rangle_H = \int_\Omega \big(h_1(x)\nabla\varphi_1\nabla\psi_1 + h_2(x)\nabla\varphi_2\nabla\psi_2\big)\,dx$$

for all $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2) \in H$.

The key in our arguments is the following lemma, which is introduced by P. Caldiroli and R. Musina [6] as the generalization of the Caffarelli–Kohn–Nirenberg inequality in [4] and [7].

Lemma 1.1. *(See [6, Proposition 2.5].) Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqq 2$. Assume that the function $h : \Omega \to [0,+\infty)$ belongs to $L^1_{loc}(\Omega)$ and satisfies the condition*

$$\lim_{x\to z}\inf |x-z|^{-\phi} h(x) > 0 \tag{1.5}$$

for all $z \in \overline{\Omega}$, where $\phi \in (0,2)$. Then there exists a constant $C_\phi > 0$ depending on ϕ such that

$$\left(\int_\Omega |\varphi|^{2^\star_\phi}\,dx\right)^{\frac{2}{2^\star_\phi}} \leqq C_\phi \int_\Omega h(x)|\nabla\varphi|^2\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$, where $2^\star_\phi = \frac{2N}{N-2+\phi}$.

By Lemma 1.1, Propositions 3.2 and 3.4 in [6] we have the following remark, which helps us to overcome the lack of compactness.

Remark 1.2. Assume that the hypotheses ($\mathbf{H_1}$) and ($\mathbf{H_2}$) are satisfied, then we conclude that

(**i**) the embedding $H \hookrightarrow L^{2^\star_\alpha}(\Omega) \times L^{2^\star_\beta}(\Omega)$ is continuous;
(**ii**) the embedding $H \hookrightarrow L^i(\Omega) \times L^j(\Omega)$ is compact for all $i \in [1, 2^\star_\alpha)$ and all $j \in [1, 2^\star_\beta)$.

Definition 1.3. We say that $w = (u, v) \in H$ is a weak solution of system (1.1) if and only if

$$\int_\Omega \big(h_1(x)\nabla u \nabla\varphi_1 + h_2(x)\nabla v \nabla\varphi_2\big)\,dx - \lambda \int_\Omega \big(f(x, u, v)\varphi_1 + g(x, u, v)\varphi_2\big)\,dx = 0$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\Omega, \mathbb{R}^2)$.

Now, we can describe our main results as follows.

Theorem 1.4. *Assume that the conditions* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *and* $(\mathbf{F_1})$ *are satisfied. Then, there exists a constant* $\underline{\lambda} > 0$ *such that for all* $\lambda < \underline{\lambda}$, *system* (1.1) *has no nontrivial weak solution.*

Theorem 1.5. *Assume that the conditions* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *and* $(\mathbf{F_1})$–$(\mathbf{F_3})$ *are satisfied. Then, there exists a constant* $\bar{\lambda} > 0$ *such that system* (1.1) *has at least two distinct, nonnegative, nontrivial weak solutions, provided that* $\lambda \geqq \bar{\lambda}$.

2. Proof of the main results

In this section, we denote by $\lambda_1(h)$ the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(h(x)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the function h satisfies all assumptions of Lemma 1.1. Then, we recall the result in [6] that $\lambda_1(h) > 0$ and is given by

$$\lambda_1(h) := \inf_{\phi \in H_0^1(\Omega, h)\setminus\{0\}} \frac{\int_\Omega h(x)|\nabla\phi|^2\,dx}{\int_\Omega |\phi|^2\,dx}. \tag{2.2}$$

Moreover, it is achieved in $H_0^1(\Omega, h)$ by a nonnegative and unique (up to multiplicative constant) function ϕ_1.

We also let λ_1 be the first eigenvalue of the following Dirichlet problem (see [22] or [24, Lemma 2.3] for $\mu(x) \equiv 1$),

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda |u|^{\gamma-1}|v|^{\delta+1} & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda |u|^{\gamma+1}|v|^{\delta-1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where the functions $h_1(x)$ and $h_2(x)$ as in $(\mathbf{H_1})$ and $(\mathbf{H_2})$, γ and δ are two positive real numbers satisfying the condition $(\mathbf{F_2})$.

Then, we have $\lambda_1 > 0$ and is given by

$$\lambda_1 = \inf_{w=(u,v)\in H\setminus\{(0,0)\}} \frac{\int_\Omega (\frac{\gamma+1}{p}h_1(x)|\nabla u|^2 + \frac{\delta+1}{q}h_2(x)|\nabla v|^2)\,dx}{\int_\Omega |u|^{\gamma+1}|v|^{\delta+1}\,dx} \tag{2.3}$$

and the associated eigenfunction $w_0 = (u_0, v_0)$ is componentwise nonnegative and is unique (up to multiplication by a nonzero scalar). Now, we are in the position to prove our main results.

Proof of Theorem 1.4. If $w = (u, v) \in H$ is a weak solution of system (1.1) then multiplying first two equations in (1.1) by u and v, respectively, integrating by parts and using $(\mathbf{F_1})$, we get

$$\begin{aligned} \int_\Omega h_1(x)|\nabla u|^2\,dx &= \lambda \int_\Omega F_u(x, u, v)u\,dx \\ &\leqq \lambda C_1 \int_\Omega |u|^{\gamma+1}|v|^{\delta+1}\,dx, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega h_2(x)|\nabla u|^2\,dx &= \lambda \int_\Omega F_v(x, u, v)u\,dx \\ &\leqq \lambda C_2 \int_\Omega |u|^{\gamma+1}|v|^{\delta+1}\,dx. \end{aligned}$$

It follows that

$$\int_\Omega \left(\frac{\gamma+1}{p}h_1(x)|\nabla u|^2 + \frac{\delta+1}{q}h_2(x)|\nabla v|^2\right)dx \leqq \lambda(C_1+C_2)\int_\Omega |u|^{\gamma+1}|v|^{\delta+1}\,dx.$$

Hence, by choosing $\underline{\lambda} = \frac{\lambda_1}{C_1+C_2}$, where λ_1 is given by (2.3), we conclude the proof of Theorem 1.4. □

In order to prove Theorem 1.5 using critical point theory, we first set $F(x,t,s) = 0$ for all t, $s < 0$, and consider for each $\lambda > 0$ the functional $\Phi_\lambda : H \to \mathbb{R}$ given by

$$\begin{aligned}\Phi_\lambda(w) &= \frac{1}{2}\int_\Omega \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right)dx - \lambda\int_\Omega F(x,u,v)\,dx \\ &= \Lambda(w) - \lambda I(w),\end{aligned} \tag{2.4}$$

where

$$\Lambda(w) = \frac{1}{2}\int_\Omega \left(h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right)dx, \tag{2.5}$$

$$I(w) = \int_\Omega F(x,u,v)\,dx \tag{2.6}$$

for all $w = (u,v) \in H$. A simple computation implies that Φ_λ is well defined and of C^1 class in H. Thus, weak solutions of (1.1) are exactly the critical points of the functional Φ_λ. We first have the following lemma.

Lemma 2.1. *The functional Φ_λ given by* (2.4) *is weakly lower semicontinuous in the space H.*

Proof. Let $\{w_m\} = \{(u_m, v_m)\}$ be a sequence that converges weakly to $w = (u,v)$ in the space $H = H_0^1(\Omega, h_1) \times H_0^1(\Omega, h_2)$. By the weak lower semicontinuity of the norms in the spaces $H_0^1(\Omega, h_1)$ and $H_0^1(\Omega, h_2)$ we deduce that

$$\liminf_{m\to\infty}\int_\Omega \left[h_1(x)|\nabla u_m|^2 + h_2(x)|\nabla v_m|^2\right]dx \geqq \int_\Omega \left[h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2\right]dx. \tag{2.7}$$

We shall show that

$$\lim_{m\to\infty}\int_\Omega F(x,u_m,v_m)\,dx = \int_\Omega F(x,u,v)\,dx. \tag{2.8}$$

Indeed, we have

$$\begin{aligned}\int_\Omega \left[F(x,u_m,v_m) - F(x,u,v)\right]dx &= \int_\Omega \nabla F\big(x, w + \theta_m(w_m - w)\big)\cdot(w_m - w)\,dx \\ &= \int_\Omega F_u\big(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v)\big)(u_m - u)\,dx \\ &\quad + \int_\Omega F_v\big(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v)\big)(v_m - v)\,dx,\end{aligned} \tag{2.9}$$

where $\theta_m = (\theta_{1,m}, \theta_{2,m})$ and $0 \leqq \theta_{1,m}(x), \theta_{2,m}(x) \leqq 1$ for all $x \in \Omega$.

Now, using $(\mathbf{F_1})$ and Hölder's inequality we conclude that

$$\begin{aligned}&\left|\int_\Omega \left[F(x,u_m,v_m) - F(x,u,v)\right]dx\right| \\ &\quad \leqq \int_\Omega \left|F_u\big(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v)\big)\right||u_m - u|\,dx \\ &\qquad + \int_\Omega \left|F_v\big(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v)\big)\right||v_m - v|\,dx\end{aligned}$$

$$\leqq C_1 \int_\Omega \left|u+\theta_{1,m}(u_m-u)\right|^\gamma \left|v+\theta_{2,m}(v_m-v)\right|^{\delta+1} |u_m-u|\,dx$$
$$+ C_2 \int_\Omega \left|u+\theta_{1,m}(u_m-u)\right|^{\gamma+1} \left|v+\theta_{2,m}(v_m-v)\right|^{\delta} |v_m-v|\,dx$$
$$\leqq C_1 \left\|u+\theta_{1,m}(u_m-u)\right\|^\gamma_{L^p(\Omega)} \left\|v+\theta_{2,m}(v_m-v)\right\|^{\delta+1}_{L^q(\Omega)} \|u_m-u\|_{L^p(\Omega)}$$
$$+ C_2 \left\|u+\theta_{1,m}(u_m-u)\right\|^{\gamma+1}_{L^p(\Omega)} \left\|v+\theta_{2,m}(v_m-v)\right\|^{\delta}_{L^q(\Omega)} \|v_m-v\|_{L^q(\Omega)}. \tag{2.10}$$

On the other hand, since $2<\gamma+1<p<2^\star_\alpha$ and $2<\gamma+1<q<2^\star_\beta$, by Remark 1.2, the sequence $\{w_m\}$ converges strongly to $w=(u,v)$ in the space $L^p(\Omega)\times L^q(\Omega)$, i.e., $\{u_m\}$ converges strongly to u in $L^p(\Omega)$ and $\{v_m\}$ converges strongly to v in $L^q(\Omega)$. Hence, it is easy to see that the sequences $\{\|u+\theta_{1,m}(u_m-u)\|_{L^p(\Omega)}\}$ and $\{\|v+\theta_{2,m}(v_m-v)\|_{L^q(\Omega)}\}$ are bounded. Thus, it follows from (2.10) that relation (2.8) holds true.

Finally, relations (2.7) and (2.8) imply that

$$\lim_{m\to\infty} \inf \Phi_\lambda(w_m) \geqq \Phi_\lambda(w) \tag{2.11}$$

and the functional Φ_λ is weakly lower semicontinuous in the space H. □

Lemma 2.2. *The functional Φ_λ given by* (2.4) *is coercive and bounded from below in the space H.*

Proof. By ($\mathbf{F_1}$), there exists $C_3>0$ such that for all $(t,s)\in\mathbb{R}^2$ and a.e. $x\in\Omega$ we deduce that

$$\left|F(x,t,s)\right| \leqq C_3|t|^{\gamma+1}|s|^{\delta+1}. \tag{2.12}$$

For real numbers p,q,γ,δ as in ($\mathbf{F_2}$), we define the number θ by

$$\theta = \frac{1}{2\max\{\frac{\gamma+1}{p},\frac{\delta+1}{q}\}} > 0. \tag{2.13}$$

Then, by ($\mathbf{F_3}$), there is a positive constant M_λ depending on λ such that for all $(t,s)\in\mathbb{R}^2$ with $|(t,s)|\geqq M_\lambda$ and for a.e. $x\in\Omega$ we get

$$F(x,t,s) \leqq \frac{\theta\lambda_1}{2\lambda}|t|^{\gamma+1}|s|^{\delta+1}, \tag{2.14}$$

where λ_1 is given by (2.3). Hence, relations (2.12) and (2.14) imply that for all $(t,s)\in\mathbb{R}^2$ and for a.e. $x\in\Omega$, it holds that

$$\lambda F(x,t,s) \leqq \frac{\theta\lambda_1}{2}|t|^{\gamma+1}|s|^{\delta+1} + C_\lambda \tag{2.15}$$

for some positive real number C_λ which depends on λ. Hence, by the definition of the functional Φ_λ we deduce that

$$\Phi_\lambda(w) \geqq \theta \int_\Omega \left[\frac{\gamma+1}{p}h_1(x)|\nabla u|^2\,dx + \frac{\delta+1}{q}h_2(x)|\nabla u|^2\right]dx - \int_\Omega \left[\frac{\theta\lambda_1}{2}|u|^{\gamma+1}|v|^{\delta+1} + C_\lambda\right]dx$$
$$\geqq \frac{\theta(\gamma+1)}{2p}\|u\|^2_{h_1} + \frac{\theta(\delta+1)}{2q}\|v\|^2_{h_2} - C_\lambda|\Omega|_N$$
$$= C_4\left(\|u\|^2_{h_1} + \|v\|^2_{h_2}\right) - C_\lambda|\Omega|_N,$$

for all $w=(u,v)\in H$, where

$$C_4 = \frac{\theta}{2}\min\left\{\frac{\gamma+1}{p},\frac{\delta+1}{q}\right\}$$

and $|\cdot|_d$ denotes the d-dimensional Lebesgue measure in $\mathbb{R}^N$, so the functional Φ_λ is coercive and bounded from below. □

Lemma 2.3. *If $w=(u,v)\in H$ is a weak solution of system* (1.1) *then $u\geqq 0$ and $v\geqq 0$ in Ω.*

Proof. Indeed, if $w=(u,v)\in H$ is a weak solution of system (1.1), then we have

$$0 = \left(D\Phi_\lambda(w), w^-\right)$$
$$= \int_\Omega \left(h_1(x)\nabla u\cdot\nabla u^- + h_2(x)\nabla v\cdot\nabla v^-\right)dx - \lambda\int_\Omega \left(F_u(x,u,v)u^- + F_v(x,u,v)v^-\right)dx$$
$$= \|u^-\|^2_{h_1} + \|v^-\|^2_{h_2},$$

where $u^-(x) = \min\{u(x), 0\}$, $v^-(x) = \min\{v(x), 0\}$ are the negative parts of u and v, respectively. Moreover, by relation (2.2) and the fact that

$$0 = \|u^-\|_{h_1}^2 \geqq \lambda_1(h_1) \int_\Omega |u^-|^2\, dx,$$

and

$$0 = \|v^-\|_{h_2}^2 \geqq \lambda_1(h_2) \int_\Omega |v^-|^2\, dx,$$

it follows that $u(x) \geqq 0$ and $v(x) \geqq 0$ for a.e. $x \in \Omega$. □

By Lemmas 2.1, 2.2 and 2.3, applying the Minimum principle (see [20, p. 4, Theorem 1.2]), the functional Φ_λ has a global minimum and thus system (1.1) admits a nonnegative weak solution $w_1 = (u_1, v_1) \in H$. The following lemma shows that the solution w_1 is not trivial provided that λ is large enough.

Lemma 2.4. *There exists a constant $\bar\lambda > 0$ such that for all $\lambda \geqq \bar\lambda$, $\inf_H \Phi_\lambda < 0$, and hence the solution $w_1 \not\equiv 0$.*

Proof. Indeed, let Ω' be a sufficiently large compact subset of Ω, a function $w_0 = (u_0, v_0) \in H$ is taken such that $u_0(x) = t_0$, $v_0(x) = s_0$ on Ω', $0 \leqslant u_0(x) \leqslant t_0$, $0 \leqslant v_0(x) \leqslant s_0$ on $\Omega\backslash\Omega'$, where t_0, s_0 are given as in ($\mathbf{F_2}$). Then we have

$$\int_\Omega F(x, u_0, v_0)\, dx \geqq \int_{\Omega'} F(x, t_0, s_0)\, dx - C_3\big(t_0^{\gamma+1} s_0^{\delta+1}\big)|\Omega\backslash\Omega'|_N > 0. \tag{2.16}$$

So, we deduce that

$$\begin{aligned}\Phi_\lambda(w_0) &= \frac{1}{2}\int_\Omega \big[h_1(x)|\nabla u_0|^2\, dx + h_2(x)|\nabla v_0|^2\big]\, dx - \lambda \int_\Omega F(x, u_0, v_0)\, dx \\ &\leqslant \frac{1}{2}\|u_0\|_{H_1}^2 + \frac{1}{2}\|v_0\|_{H_2}^2 - \lambda\bigg(\int_{\Omega'} F(x, t_0, s_0)\, dx - C_3\big(t_0^{\gamma+1} s_0^{\delta+1}\big)|\Omega\backslash\Omega'|_N\bigg).\end{aligned}$$

Hence, if Ω' is large enough, there exists $\bar\lambda$ such that for all $\lambda \geqslant \bar\lambda$ we have $\Phi_\lambda(w_0) < 0$, thus $w_1 \not\equiv 0$. Moreover, $\Phi_\lambda(w_1) < 0$ for all $\lambda \geqslant \bar\lambda$. □

In the next parts, we shall show the existence of the second weak solution $w_2 = (u_2, v_2) \in H$ $(w_2 \neq w_1)$ of system (1.1) by applying the Mountain pass theorem in [1]. To this purpose, we first show that for all $\lambda \geqq \bar\lambda$, the functional Φ_λ has the geometry of the Mountain pass theorem.

Lemma 2.5. *There exist a constant $\rho \in (0, \|w_1\|_H)$ and a constant $r > 0$ such that $\Phi_\lambda(w) \geqq r$ for all $w \in H$ with $\|w\|_H = \rho$.*

Proof. For each $w = (u, v) \in H$ we set

$$\Omega_w = \big\{x \in \Omega\colon\ \big|u(x)\big|^p + \big|v(x)\big|^q > \eta\big\}, \tag{2.17}$$

where p, q and η are given as in ($\mathbf{F_2}$). Then, we have $F(x, u(x), v(x)) \leqq 0$ on $\Omega\backslash\Omega_w$. Hence, using Young's and Hölder's inequalities, relations (2.2) and (2.12) we get

$$\begin{aligned}\int_{\Omega_w} F(x, u, v)\, dx &\leqq C_3 \int_{\Omega_w} |u|^{\gamma+1}|v|^{\delta+1}\, dx \\ &\leqq C_3 \int_{\Omega_w} \bigg[\frac{\gamma+1}{p}|u|^p + \frac{\delta+1}{q}|v|^q\bigg]\, dx \\ &\leqq C_3\frac{\gamma+1}{p}\bigg(\int_{\Omega_w} |u|^{p'}\, dx\bigg)^{\frac{p}{p'}} |\Omega_w|^{1-\frac{p}{p'}} + C_3\frac{\delta+1}{q}\bigg(\int_{\Omega_w} |v|^{q'}\, dx\bigg)^{\frac{q}{q'}} |\Omega_w|^{1-\frac{q}{q'}} \\ &\leqq C_5\frac{\gamma+1}{p}\|u\|_{h_1}^p |\Omega_w|^{1-\frac{p}{p'}} + C_5\frac{\delta+1}{q}\|v\|_{h_2}^q |\Omega_w|^{1-\frac{q}{q'}},\end{aligned} \tag{2.18}$$

where $p' \in (p, 2^\star_\alpha)$ and $q' \in (q, 2^\star_\beta)$. Thus,

$$\begin{aligned}\Phi_\lambda(u,v) &\geqq \frac{1}{2}\|u\|_{h_1}^2 + \frac{1}{2}\|v\|_{h_2}^2 - \lambda \int_{\Omega_w} F(x,u,v)\,dx \\ &\geqq \|u\|_{h_1}^2\left(\frac{1}{2} - \lambda C_5 \frac{\gamma+1}{p}\|u\|_{h_1}^{p-2}|\Omega_w|^{1-\frac{p}{p'}}\right) + \|v\|_{h_2}^2\left(\frac{1}{2} - \lambda C_5 \frac{\delta+1}{q}\|v\|_{h_2}^{q-2}|\Omega_w|^{1-\frac{q}{q'}}\right). \qquad (2.19)\end{aligned}$$

Since $p > \gamma+1 > 2$ and $q > \delta+1 > 2$, in order to prove Lemma 2.5, it is enough to show that

$$|\Omega_w| \to 0 \quad \text{as} \quad \|w\|_H = \|u\|_{h_1} + \|v\|_{h_2} \to 0.$$

Indeed, let $\epsilon > 0$ be arbitrary, we choose $\Omega_\epsilon \subset \Omega$ a compact subset, large enough such that $|\Omega\backslash\Omega_\epsilon| < \epsilon$ and denote by $\Omega_{w,\epsilon} := \Omega_w \cap \Omega_\epsilon$. Then, by Remark 1.2, it is clear that for all $w = (u,v) \in H$ we deduce that

$$\begin{aligned}\|u\|_{h_1}^p + \|v\|_{h_2}^q &\geqq C_p^p \int_\Omega |u|^p\,dx + C_q^q \int_\Omega |v|^q\,dx \\ &\geqq \min\{C_p^p, C_q^q\} \int_{\Omega_{w,\epsilon}} \left(|u|^p + |v|^q\right)dx \\ &\geqq \min\{C_p^p, C_q^q\}\eta|\Omega_{w,\epsilon}|, \qquad (2.20)\end{aligned}$$

where C_p and C_q denote by the best constants in the embeddings $H_0^1(\Omega, h_1) \hookrightarrow L^p(\Omega)$ and $H_0^1(\Omega, h_2) \hookrightarrow L^q(\Omega)$, respectively, and η as in ($\mathbf{F_2}$).

Letting $\|w\|_H \to 0$ we deduce that $\|u\|_{h_1} \to 0$ and $\|v\|_{h_2} \to 0$. Combining these with the above information we conclude that $|\Omega_{w,\epsilon}| \to 0$. Since $\Omega_w \subset \Omega_{w,\epsilon} \cup \Omega\backslash\Omega_\epsilon$ we have

$$|\Omega_w| \leqq |\Omega_{w,\epsilon}| + \epsilon$$

with $\epsilon > 0$ is arbitrary. Thus, $|\Omega_w| \to 0$ as $\|w\|_H \to 0$. The proof of the lemma is complete. □

Lemma 2.6. *The functional Φ_λ given by* (2.4) *satisfies the Palais–Smale condition in H.*

Proof. By Lemma 2.2, we deduce that Φ_λ is coercive on H. Let $\{w_m\} = \{(u_m, v_m)\}$ be a Palais–Smale sequence for the functional Φ_λ in H, i.e.

$$\left|\Phi_\lambda(u_m)\right| \leqq C_6 \quad \text{for all } m, \qquad D\Phi_\lambda(u_m) \to 0 \quad \text{in } H^{-1} \text{ as } m \to \infty, \qquad (2.21)$$

where H^{-1} is the dual space of H.

Since Φ_λ is coercive on H, relation (2.21) implies that the sequence $\{w_m\}$ is bounded in H. Since H is a Hilbert space, there exists $w = (u,v) \in H$ such that, passing to a subsequence, still denoted by $\{w_m\}$, it converges weakly to w in H. Hence, $\{\|w_m - w\|\}$ is bounded. This and (2.21) imply that $D\Phi_\lambda(w_m)(w_m - w)$ converges to 0 as $m \to \infty$. Using the condition ($\mathbf{F_1}$) combined with Hölder's inequality we conclude that

$$\begin{aligned}\int_\Omega \left|F_u(x,u_m,v_m)\right||u_m - u|\,dx &\leqq C_1 \int_\Omega |u_m|^\gamma |v_m|^{\delta+1}|u - u_m|\,dx \\ &\leqq C_1\|u_m\|_{L^p(\Omega)}^\gamma \|v_m\|_{L^q(\Omega)}^{\delta+1}\|u_m - u\|_{L^p(\Omega)}, \qquad (2.22)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \left|F_v(x,u_m,v_m)\right||v_m - v|\,dx &\leqq C_2 \int_\Omega |u_m|^{\gamma+1} |v_m|^{\delta}|u - u_m|\,dx \\ &\leqq C_2\|u_m\|_{L^p(\Omega)}^{\gamma+1} \|v_m\|_{L^q(\Omega)}^{\delta}\|v_m - v\|_{L^q(\Omega)}. \qquad (2.23)\end{aligned}$$

It follows from relations (2.22) and (2.23) that

$$\begin{aligned}\left|DI(w_m)(w_m - w)\right| &= \left|\int_\Omega \left[F_u(x,u_m,v_m)(u_m - u) + F_v(x,u_m,v_m)(v_m - v)\right]dx\right| \\ &\leqq C_1\|u_m\|_{L^p(\Omega)}^\gamma \|v_m\|_{L^q(\Omega)}^{\delta+1}\|u_m - u\|_{L^p(\Omega)} + C_2\|u_m\|_{L^p(\Omega)}^{\gamma+1} \|v_m\|_{L^q(\Omega)}^{\delta}\|v_m - v\|_{L^q(\Omega)}\end{aligned}$$

where the functional I is given by (2.6). Therefore, we deduce by Remark 1.2 that

$$\lim_{m\to\infty} DI(w_m)(w_m - w) = 0. \tag{2.24}$$

Combining this with (2.21) and the fact that

$$D\Lambda(w_m)(w_m - w) = D\Phi_\lambda(w_m)(w_m - w) + DI(w_m)(w_m - w)$$

imply that

$$\lim_{m\to\infty} D\Lambda(w_m)(w_m - w) = 0, \tag{2.25}$$

where the functional Λ is given by (2.5).

Hence, by the convexity of the functional Λ, we have

$$\begin{aligned}\Lambda(w) - \lim_{m\to\infty} \sup \Lambda(w_m) &= \lim_{m\to\infty} \inf\bigl(\Lambda(w) - \Lambda(w_m)\bigr)\\ &\geqq \lim_{m\to\infty} D\Lambda(w_m)(w - w_m) = 0\end{aligned} \tag{2.26}$$

and the weak lower semicontinuity of Λ implies that

$$\lim_{m\to\infty} \Lambda(w_m) = \Lambda(w). \tag{2.27}$$

We now assume by contradiction that $\{w_m\}$ does not converge strongly to w in H, then there exist a constant $\epsilon > 0$ and a subsequence of $\{w_m\}$, still denoted by $\{w_m\}$, such that $\|w_m - w\| \geqq \epsilon$. We have

$$\frac{1}{2}\Lambda(w) + \frac{1}{2}\Lambda(w_m) - \Lambda\left(\frac{w_m + w}{2}\right) = \frac{1}{4}\|w_m - u\|^2 \geqq \frac{1}{4}\epsilon^2. \tag{2.28}$$

Letting $m \to \infty$, relation (2.28) gives

$$\lim_{m\to\infty} \sup \Lambda\left(\frac{w_m + w}{2}\right) \leqq \Lambda(w) - \frac{1}{4}\epsilon^2. \tag{2.29}$$

We remark that the sequence $\{\frac{w_m+w}{2}\}$ also converges weakly to w in H. So, we get

$$\Lambda(w) \leqq \lim_{m\to\infty} \inf \Lambda\left(\frac{w_m + w}{2}\right), \tag{2.30}$$

which contradicts (2.29). Therefore, $\{w_m\}$ converges strongly to w in H and the functional Φ_λ satisfies the Palais–Smale condition in H. $\square$

Proof of Theorem 1.5. By Lemmas 2.1–2.4, system (1.1) admits a nonnegative, nontrivial weak solution $w_1 = (u_1, v_1)$ as the global minimizer of Φ_λ. Set

$$\bar{c} := \inf_{\chi\in\boldsymbol{\Gamma}} \max_{w\in\chi([0,1])} \Phi_\lambda(w), \tag{2.31}$$

where $\boldsymbol{\Gamma} := \{\chi \in C([0,1], H):\ \chi(0) = 0,\ \chi(1) = w_1\}$.

Lemmas 2.5–2.6 show that all assumptions of the Mountain pass theorem in [1] are satisfied, $\Phi_\lambda(w_1) = \Phi_\lambda(w_1) < 0$ and $\|w_1\|_H > \rho$. Then, $\bar{c}$ is a critical value of Φ_λ, i.e. there exists $w_2 = (u_2, v_2) \in H$ such that $D\Phi_\lambda(w_2)(\varphi) = 0$ for all $\varphi \in H$ or w_2 is a weak solution of (1.1). Moreover, w_2 is not trivial and $w_2 \not\equiv w_1$ since $\Phi_\lambda(w_2) = \bar{c} > 0 > \Phi_\lambda(w_1)$. Theorem 1.5 is completely proved. $\square$

3. Final comments

In this section, we make some comments regarding extensions of system (1.1). While uniform elliptic problems (equations and systems) are intensively studied in the last decades, the degenerate elliptic problems still contain some unknown things. For problem (1.1), the reader may be interested in some further directions of research as follows:

1. In the hypothesis ($\mathbf{F_1}$), we require that $\gamma, \delta > 1$. This condition helps us to show that the functional has the geometry of the Mountain pass theorem [1] (see Lemma 2.5). What happens if we only require that $\gamma, \beta > 0$? In this paper, we have not considered the problem with critical exponent, i.e., $\gamma = 2_\alpha^\star - 1$ and $\delta = 2_\beta^\star - 1$ yet (see [21]).
2. May Theorem 1.4 and Theorem 1.5 be valid for the discontinuous nonlinearities as in [24]?
3. Finally, the reader may study the existence of sign-changing solutions for system (1.1) (see [5, Theorems 2.12 and 2.13]).

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